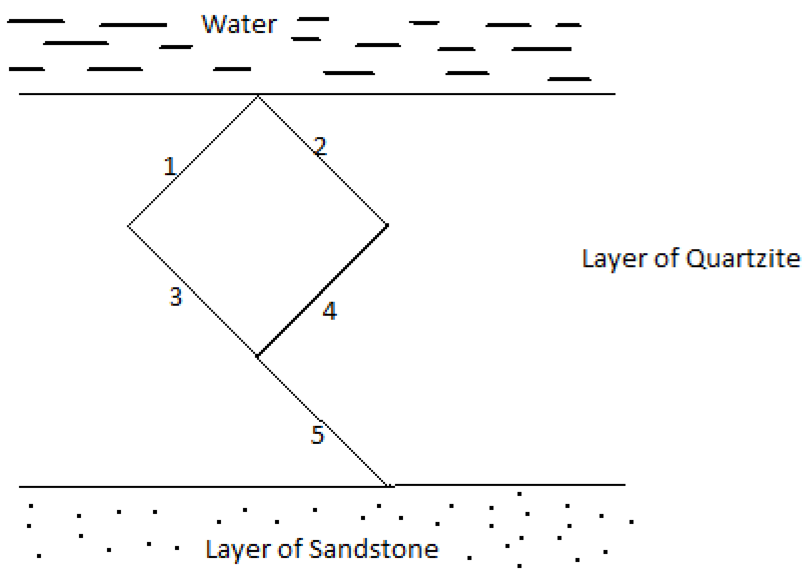
Probability II: B. Math (Hons.) I
Academic Year 2022-23, Second Semester
Backpaper Exam
Total Marks = 100 Duration: 3 hours

- **Show all your work and write explanations when needed. If you are using a result stated and/or proved in class, please quote it correctly.**
- **You are NOT allowed to use class notes, books, homework solutions, list of theorems, formulas etc.**

1. Consider the following schematic diagram of a drainage network model (as described in class), where each of the five paths is open with probability $p = 0.5$ and the paths behave independently of each other.



   (a) (5 marks) Let X be the number of open paths and Y be the indicator that water can pass through the layer of quartzite to the layer of sandstone. Find the conditional probability mass function of X given $Y = 1$.
   (b) (5 marks) Compute $E(X|Y = 1)$.
   (c) (5 marks) Compute $Var(X|Y = 1)$.

Please Turn Over

2. A continuous random vector (X, Y) has a joint probability density function given by

$$f_{X,Y}(x, y) = c \quad \text{if} \ \ 0 < x + y < 2 \ \text{ and } \ 0 < x - y < 2,$$

where c is a positive constant. Define $U := (X + Y)/2$ and $V := (X - Y)/2$.

(a) (2 marks) Find c.

(b) (3 marks) Are X and Y independent? Please justify your answer.

(c) (5+5 = 10 marks) Calculate the conditional probability density functions of X given Y and Y given X.

(d) (10 marks) Calculate a joint probability density function of U and V.

(e) (3 marks) Are U and V independent? Please justify your answer.

(f) (2 marks) Find the marginal distributions of U and V.

3. ((3+4)+8 = 15 marks) Suppose $r(\geq 2)$ distinguishable umbrellas are distributed at random among $n(\geq 3)$ mathematicians. Let X be the number of mathematician(s) who get no umbrella and Y be the number of mathematician(s) who get exactly one umbrella. Compute $E(X)$ and $Var(X)$. Show that

$$Cov(X, Y) = \frac{r(n-1)(n-2)^{r-1}}{n^{r-1}} - \frac{r(n-1)^{2r-1}}{n^{2r-2}}.$$

4. (7+(4+4) = 15 marks) If $X \sim Poi(\lambda)$ and $Y \sim Poi(\mu)$ are independent, then find the conditional distribution of X given $X + Y$. In this case, compute $E(X|X + Y)$ and verify (by direct computation) that $E\big[E(X|X + Y)\big] = E(X)$.

5. Suppose X and Y are jointly normal (i.e., bivariate normal) with $E(X) = E(Y) = 0$, $Var(X) = Var(Y) = 1$ and $Corr(X, Y) = \rho \in (-1, 1)$.

(a) (4 marks) Show that $(X, Y)^T \stackrel{d}{=} \big(Z,\ \rho Z + \sqrt{1-\rho^2}\, W\big)^T$, where $Z, W \stackrel{iid}{\sim} N(0, 1)$.

(b) (4 marks) Using (a) or otherwise, state an algorithm to simulate the random vector $(X, Y)^T$ using $U, V \stackrel{iid}{\sim} Unif(0, 1)$. Just state the method. No proof is required.

(c) (7 marks) Using (a) or otherwise, compute, with full justification, the correlation coefficient between X^2 and Y^2.

6. (10 marks) Suppose $(X_1, X_2, X_3, X_4, X_5) \sim Dir(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5; \alpha_6)$ (the notation is as used in the class), where $\alpha_1, \alpha_2, \ldots, \alpha_6$ are strictly positive parameters. Compute, with full justification, a probability density function of $X_1 + X_2 + \cdots + X_5$.